

ON CLOSED LIE IDEALS OF CERTAIN TENSOR PRODUCTS OF C^* -ALGEBRAS

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ABSTRACT. For a simple C^* -algebra A and any other C^* -algebra B , it is proved that every closed ideal of $A \otimes^{\min} B$ is a product ideal if either A is exact or B is nuclear. Closed commutator of a closed ideal in a Banach algebra whose every closed ideal possesses a quasi-central approximate identity is described in terms of the commutator of the Banach algebra. If α is either the Haagerup norm, the operator space projective norm or the C^* -minimal norm, then this allows us to identify all closed Lie ideals of $A \otimes^{\alpha} B$, where A and B are simple, unital C^* -algebras with one of them admitting no tracial functionals, and to deduce that every non-central closed Lie ideal of $B(H) \otimes^{\alpha} B(H)$ contains the product ideal $K(H) \otimes^{\alpha} K(H)$. Closed Lie ideals of $A \otimes^{\min} C(X)$ are also determined, A being any simple unital C^* -algebra with at most one tracial state and X any compact Hausdorff space. And, it is shown that closed Lie ideals of $A \otimes^{\alpha} K(H)$ are precisely the product ideals, where A is any unital C^* -algebra and α any completely positive uniform tensor norm.

1. Introduction

A complex associative algebra A inherits a canonical Lie algebra structure given by the bracket $A \times A \ni (x, y) \mapsto [x, y] := xy - yx \in A$ and a subspace L of A is said to be a *Lie ideal* if $[a, x] \in L$ for all $a \in A$ and $x \in L$.

Analysis of ideal structures of various tensor products of operator algebras has been an important project and a good deal of work has been done in this direction - see, for instance, [3, 10, 14, 15, 12, 16, 24]. On the other hand, there also exists an extensive literature devoted towards the study of Lie ideals, directly as well as through ideals of the algebra, in pure as well as Banach and operator algebras - see [6, 9, 17, 18, 19, 20] and the references therein.

The analysis of closed Lie ideals in operator algebras is primarily motivated by the evident relationship between commutators, projections and closed Lie ideals in C^* -algebras. For instance, Pedersen ([21, Lemma 1]) showed that the closed subspace $L(\mathcal{P})$ and the C^* -subalgebra $A(\mathcal{P})$ generated by the set of projections \mathcal{P} of a C^* -algebra A are both closed Lie ideals of A ; and, moreover, if A is simple with a non-trivial projection and if A has at most one tracial state then $A = L(\mathcal{P})$, i.e., the span of the projections is dense in A ([21, Corollary 4]).

However, unlike the ideals of tensor products of operator algebras, not much is known about the closed Lie ideals of various tensor products of operator algebras. Among the few

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known results in this direction, Marcoux [17], in 1995, proved that for a UHF C^* -algebra A , a subspace L of $A \otimes^{\min} C(X)$ is a closed Lie ideal if and only if

$$L = \overline{sl(A) \otimes J} + \mathbb{C}I \otimes S$$

for some closed ideal J and some closed subspace S in $C(X)$, where $sl(A) := \{a \in A : tr_A(a) = 0\}$ with respect to the unique faithful tracial state tr_A on A . Then, in 2008, relying heavily on the Lie ideal structure of tensor products of pure algebras, Brešar et al., in [6], proved that for a unital Banach algebra A , the closed Lie ideals of $A \otimes^{\min} K(H)$, of the Banach space projective tensor product $A \otimes^{\gamma} K(H)$ and of the Banach space injective tensor product $A \otimes^{\lambda} K(H)$ (if it is a Banach algebra) are precisely the closed ideals.

In this article, we focus on analyzing the (closed) ideal and Lie ideal structures of certain tensor products of C^* -algebras. Here is a quick overview of the structure of this paper.

In Section 2, we generalize a characterization (of [9, 17]) for closed Lie ideals via invariance under unitaries in a simple unital C^* -algebra containing non-trivial projections and admitting at most one tracial state. Then, in Section 3, following the footsteps of [3, 15], for a simple C^* -algebra A and any C^* -algebra B we discuss the ideal structure of $A \otimes^{\min} B$ when A is exact or B is nuclear.

Section 4 is the key part of this article. Starting with the analysis of closed commutators of closed ideals, we move on to obtain a generalization (see Corollary 4.7) of a characterization of closed Lie ideals in C^* -algebras given by Brešar et al. [6] to Banach algebras in which sufficiently many closed ideals possess *quasi-central approximate identities*. Using these, when α is either the Haagerup norm, the operator space projective norm or the C^* -minimal norm, we identify all closed Lie ideals of $A \otimes^{\alpha} B$, where A and B are simple, unital C^* -algebras with one of them admitting no tracial functionals, and, deduce that $B(H) \otimes^{\alpha} B(H)$ has only one non-zero *central* Lie ideal, namely, $\mathbb{C}(1 \otimes 1)$, whereas every *non-central* closed Lie ideal contains the product ideal $K(H) \otimes^{\alpha} K(H)$.

In Section 5, we basically show that the techniques of Marcoux and Brešar et al. can be applied to obtain an analogy to Marcoux's result ([17]) that determines the structure of Lie ideals of $A \otimes^{\min} C(X)$, where A is any simple unital C^* -algebra with at most one tracial state and X is any compact Hausdorff space. And, finally, in Section 6, applying a deep result of Brešar et al. [6], we deduce that closed Lie ideals of $A \otimes^{\alpha} K(H)$ are precisely the product ideals, where A is a unital C^* -algebra and α a *completely positive uniform tensor norm*.

2. CLOSED LIE IDEALS OF SIMPLE UNITAL C^* -ALGEBRAS

In order to maintain distinction between algebraic and topological simplicity, we shall say that a Banach algebra is *topologically simple* if it does not contain any non-trivial closed two sided ideal in it. However, since maximal ideals are closed and every proper ideal is contained in a maximal ideal in a unital Banach algebra, it is easily seen that the two notions are same for unital Banach algebras.

Recall that, a *tracial state* φ on a C^* -algebra A is a positive linear functional of norm one satisfying $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$. If A is unital, then a tracial state is unital, i.e., $\varphi(1) = \|\varphi\| = 1$. The collection of tracial states on A is denoted by $\mathcal{T}(A)$.

Note that, for each $\varphi \in \mathcal{T}(A)$, $\ker(\varphi)$ is clearly a closed Lie ideal in A of co-dimension 1 and contains the closed *commutator Lie ideal* $[A, A]$. In particular, if $\mathcal{T}(A) \neq \emptyset$, then $sl(A) := \cap \{\ker(\varphi) : \varphi \in \mathcal{T}(A)\}$ is also a closed Lie ideal in A and contains $[A, A]$. Cuntz and Pederson ([7]) proved that they are, in fact, equal.

THEOREM 2.1. ([7, Theorem 2.9], [22, Theorem 1]) *Let A be a C^* -algebra. Then, the following hold:*

- (1) $\overline{[A, A]} = \begin{cases} sl(A) & \text{if } \mathcal{T}(A) \neq \emptyset, \text{ and} \\ A & \text{if } \mathcal{T}(A) = \emptyset. \end{cases}$
- (2) *If A is unital and $\mathcal{T}(A) = \emptyset$, then $[A, A] = A$.*

It turns out that $\overline{[A, A]}$ is the only non-trivial closed Lie ideal for a large class of C^* -algebras. The following identifications of Lie ideals were made in [19, Theorem 2.5] and [6, Proposition 5.23], and we will require this list in our discussions ahead.

PROPOSITION 2.2. ([19, 6]) *Let A be a simple unital C^* -algebra.*

- (1) *If A has no tracial states, then the only Lie ideals of A are $\{0\}$, $\mathbb{C}1$ and A .*
- (2) *If A has a unique tracial state, then the only closed Lie ideals of A are $\{0\}$, $\mathbb{C}1$, $sl(A)$ and A .*

COROLLARY 2.3. *If M is a II_1 -factor or an I_n -factor, then the only (uniformly) closed Lie ideals of M are $\{0\}$, $\mathbb{C}1$, $sl(M)$ and M .*

Proof. A II_1 -factor or an I_n -factor is algebraically simple because it is a simple unital C^* -algebra - see [4, III.1.7.11]. Moreover, it has a unique tracial state - see [4, III.2.5.7]. \square

Fong, Miers and Sourour ([9, Theorem 1]) and Marcoux ([17, Theorem 2.12]) characterized closed Lie ideals of $B(H)$ and of a UHF C^* -algebra, respectively, through invariance under unitary conjugation. Note that $B(H)$ admits no tracial states and a UHF C^* -algebra admits a unique tracial state and both are spanned by their projections ([18], [19, Theorem 4.6]). Imitating the original proofs, we obtain the following generalization of above characterization.

PROPOSITION 2.4. *Let A be a simple unital C^* -algebra with at most one tracial state and suppose it contains a non-trivial projection. Let L be a closed subspace of A . Then the following are equivalent:*

- (1) *L is a Lie ideal.*
- (2) *$s^{-1}Ls \subseteq L$ for all invertible elements s in A .*
- (3) *$u^*Lu \subseteq L$ for all unitaries u in A .*

Proof. By Proposition 2.2, (1) \Rightarrow (2) is a straight forward verification on the possible list of closed Lie ideals. The implication (2) \Rightarrow (3) is obvious.

In order to show (3) \Rightarrow (1), note that for every projection $p \in A$, $u := p + i(1 - p)$ is a unitary and for $l \in L$,

$$[p, l] = \frac{(u^*lu - ulu^*)}{2i} \in L.$$

Then, since A is a simple unital C^* -algebra with either no tracial states or a unique tracial state and contains a non-trivial projection, the projections span a dense subspace of A ([21, Corollary 7]), and we are done. \square

We now show that the analogue of Proposition 2.4 does not hold in Banach algebras. Recall (from [6]) that a *tracial functional* on a Banach algebra A is a non-zero continuous linear functional φ satisfying $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$. The collection of tracial functionals on A is denoted by $\mathcal{TF}(A)$. By Hahn-Banach Theorem, we easily see that $\overline{[A, A]} = A$ if and only if $\mathcal{TF}(A) = \emptyset$.

In a unital Banach algebra A , the set of its unitaries is defined as $U(A) = \{u \in GL(A) : \|u\| = 1 = \|u^{-1}\|\}$. If A is a unital C^* -algebra, then clearly $\{u \in A : uu^* = 1 = u^*u\} \subseteq U(A)$ and for $u \in U(A)$, considering $A \subseteq B(H)$ for some Hilbert space H , we see that $\|\xi\| = \|u^{-1}u(\xi)\| \leq \|u(\xi)\| \leq \|\xi\|$ for all $\xi \in H$, so that u is an isometry. In particular, it follows that for a unital C^* -algebra A , both definitions give the same set, i.e., $U(A) = \{u \in A : uu^* = 1 = u^*u\}$.

REMARK 2.5. For any two unital C^* -algebras A and B , it is known ([13, Corollary 2]) that $U(A \otimes^h B) = \{u \otimes v : u \in U(A), v \in U(B)\}$, where \otimes^h is the Haagerup tensor product (see [8]). By a result of Fack (see [18, Theorem 2.16]), the Cuntz algebra \mathcal{O}_2 is spanned by its commutators and, therefore, it has no tracial functionals. Further, since $\|\cdot\|_h$ is cross norm (see [8]), $\mathcal{O}_2 \ni x \rightarrow x \otimes 1 \in \mathcal{O}_2 \otimes^h \mathcal{O}_2$ is an isometric homomorphism, so the Banach algebra $\mathcal{O}_2 \otimes^h \mathcal{O}_2$ does not have any tracial functionals, as well. Also, since \mathcal{O}_2 is a simple C^* -algebra, $\mathcal{O}_2 \otimes^h \mathcal{O}_2$ is a topologically simple Banach algebra, by [3, Theorem 5.1]. By above decomposition of unitaries, $\mathcal{O}_2 \otimes \mathbb{C}1$ is invariant under conjugation by unitaries in $\mathcal{O}_2 \otimes^h \mathcal{O}_2$ but it is easily seen that it is not a Lie-ideal.

On similar lines, for any infinite dimensional Hilbert space H , it can also be seen that $K(H) \otimes \mathbb{C}1$ is invariant under conjugation by unitaries in $B(H) \otimes^h B(H)$ but is not a Lie ideal. These observations also illustrate that tensor product of two Lie ideals need not be a Lie ideal.

REMARK 2.6. Unlike the above decomposition of unitaries in the Banach algebra $\mathcal{O}_2 \otimes^h \mathcal{O}_2$, the unitaries in the C^* -algebra $\mathcal{O}_2 \otimes^{\min} \mathcal{O}_2$ do not decompose as elementary tensors. Indeed, since $\mathcal{O}_2 \otimes^{\min} \mathcal{O}_2$ is a simple (see [24]), unital C^* -algebra and has no tracial states, by [6, Proposition 5.23] or Theorem 4.16 below, its only closed Lie ideals are $\{0\}$, $\mathbb{C}(1 \otimes 1)$ and itself. Since \mathcal{O}_2 contains non-trivial projections, so does $\mathcal{O}_2 \otimes^{\min} \mathcal{O}_2$; therefore, by Proposition 2.4, $\mathcal{O}_2 \otimes \mathbb{C}1$ is not invariant under conjugation by unitaries. In particular, not every unitary in $\mathcal{O}_2 \otimes^{\min} \mathcal{O}_2$ can be expressed as an elementary tensor $u \otimes v$ for unitaries u and v in \mathcal{O}_2 .

3. CLOSED IDEALS OF $A \otimes^{\min} B$

Let A and B be C^* -algebras and suppose A is topologically simple. If α is either the Haagerup tensor product or the operator space projective tensor product, then by [3, Proposition 5.2], and by [15, Theorem 3.8], it is known that every closed ideal of the Banach algebra $A \otimes^\alpha B$ is a product ideal of the form $A \otimes^\alpha J$ for some closed ideal J in B .

In general, not much is known about the ideal structure of the C^* -minimal tensor product. However, the (Zorn's Lemma) technique used in above ideal structures can be applied to analyze the ideals of $A \otimes^{\min} B$ under some additional hypothesis, which we demonstrate below.

THEOREM 3.1. *Let A and B be C^* -algebras where A is topologically simple. If either A is exact or B is nuclear, then every closed ideal of the C^* -algebra $A \otimes^{\min} B$ is a product ideal of the form $A \otimes^{\min} J$ for some closed ideal J in B .*

Proof. Let I be a non-zero closed ideal in $A \otimes^{\min} B$. Consider the collection

$$\mathcal{F} := \{J \subseteq B : J \text{ is a closed ideal in } B \text{ and } A \otimes^{\min} J \subseteq I\}.$$

By [3, Proposition 4.5], I contains a non-zero elementary tensor, say, $a \otimes b$. If K and J are the non-zero closed ideals in A and B generated by a and b , respectively, then by simplicity of A , we have $K = A$ and $A \otimes^{\min} J \subseteq I$. In particular, $\mathcal{F} \neq \emptyset$.

Note that, by injectivity of \otimes^{\min} and the fact that a finite sum of closed ideals is closed in a C^* -algebra, it is easily seen that $A \otimes^{\min} (\sum_i J_i) = \sum_i (A \otimes^{\min} J_i)$ for any finite collection of closed ideals $\{J_i\}$ in B . So, with respect to the partial order given by set inclusion, every chain $\{J_i : i \in \Lambda\}$ in \mathcal{F} has an upper bound, namely, the closure of the ideal $\{\sum_{\text{finite}} x_i : x_i \in J_i\}$ in \mathcal{F} , implying thereby that there exists a maximal element, say J , in \mathcal{F} .

We will show that $A \otimes^{\min} J = I$. Consider the map $\text{Id} \otimes^{\min} \pi : A \otimes^{\min} B \rightarrow A \otimes^{\min} (B/J)$. If A is exact, then by definition of exactness, its kernel is $A \otimes^{\min} J$; and, if B is nuclear, then so are J and B/J and it is known (see [4, 10]) that the sequence

$$0 \rightarrow A \otimes^{\max} J \rightarrow A \otimes^{\max} B \rightarrow A \otimes^{\max} (B/J) \rightarrow 0$$

is always exact and, therefore, we obtain

$$\ker(\text{Id} \otimes^{\min} \pi) = \ker(\text{Id} \otimes^{\max} \pi) = A \otimes^{\max} J = A \otimes^{\min} J.$$

Since $\text{Id} \otimes^{\min} \pi$ is a surjective $*$ -homomorphism, $\tilde{I} := (\text{Id} \otimes^{\min} \pi)(I)$ is a closed ideal in $A \otimes^{\min} (B/J)$. It is now sufficient to show that this is the zero ideal. If $\tilde{I} \neq 0$, then, again by [3, Proposition 4.5], \tilde{I} contains a non-zero elementary tensor, say, $a \otimes (b + J)$. Let K be the closed ideal in B generated by b . Since A is simple, it equals the closed ideal generated by a and we obtain $A \otimes^{\min} K \subseteq I$, a contradiction to the maximality of J as $A \otimes^{\min} K$ is not contained in $A \otimes^{\min} J$. \square

4. IDEALS WITH QUASI-CENTRAL APPROXIMATE IDENTITIES AND THEIR CLOSED COMMUTATORS

We first recall some definitions and notations from [17, 6]. Every subspace of $Z(A)$, the center of an associative algebra A , is clearly a Lie ideal in A and is called a *central Lie ideal*. For subspaces X and Y of A ,

$$[X, Y] := \text{span}\{[x, y] : x \in X, y \in Y\} \quad \text{and} \quad XY := \text{span}\{xy : x \in X, y \in Y\}.$$

If L and M are Lie ideals in A then so is $[L, M]$. For a subspace S of A , consider the subspace

$$N(S) := \{x \in A : [x, a] \in S \text{ for all } a \in A\}.$$

If L is a Lie ideal then $N(L)$ is a subalgebra as well as a Lie ideal of A ([6, Proposition 2.2]). Note that if I is an ideal in A , then any subspace L of A *embraced by* I , i.e., satisfying $[I, A] \subseteq L \subseteq N([I, A])$, is a Lie ideal in A . In fact, Brešar et al. [6, §5] showed that a closed subspace L of a C^* -algebra A is a Lie ideal if and only if it is *topologically embraced* by a closed ideal I in A , i.e.,

$$\overline{[I, A]} \subseteq L \subseteq N(\overline{[I, A]}).$$

We show below (see Corollary 4.7) that this characterization generalizes to Banach algebras whose every closed ideal possesses a *quasi-central approximate identity*. Examples of such Banach algebras (which are not C^* -algebras) will be illustrated in Section 4.1.

For a closed ideal I in a C^* -algebra A , it is known ([20, Lemma 1] and [6, Proposition 5.25]) that

$$\overline{[I, I]} = \overline{[I, A], A]} = \overline{[I, A]} = I \cap \overline{[A, A]}.$$

Miers (in [20]) mentions that the third equality was due to Bunce and gives a proof using *quasi-central approximate identity*, and the other two equalities were proved by Brešar et al. using techniques of von Neumann algebras. We generalize this result to ideals in Banach

algebras with quasi-central approximate identities. The proof given here borrows ideas from [20, 23] and does not involve von Neumann algebra tools.

DEFINITION 4.1. [3] If I is an ideal in a Banach algebra A , then a net $\{e_\lambda\}$ in I is said to be a quasi-central approximate identity for I in A if

- (1) $\sup_\lambda \|e_\lambda\| < \infty$, and
- (2) $\lim_\lambda \|xe_\lambda - x\| = \lim_\lambda \|e_\lambda x - x\| = \lim_\lambda \|e_\lambda a - ae_\lambda\| = 0$ for all $x \in I$ and $a \in A$.

It is known that all ideals (not necessarily closed) in C^* -algebras possess quasi-central approximate identities ([1, Theorem 3.2] and [2, Theorem 1]).

The following equalities between commutators of ideals will be required ahead in a characterization of closed Lie ideals (see [20, Lemma 1], [6, Proposition 5.25] and [23, Lemma 1.4] for ideals in C^* -algebras).

LEMMA 4.2. *Let I be a closed ideal in a Banach algebra A . If I admits a quasi-central approximate identity in A , then*

$$(4.1) \quad \overline{[I, I]} = \overline{[I, A]} = I \cap \overline{[A, A]}.$$

In particular, $N(\overline{[I, A]}) = N(I)$ and, if A has no tracial functionals, then $\overline{[I, A]} = I$. Moreover, if the closed ideal $J := \overline{\text{Id}[I, A]}$ also contains a quasi-central approximate identity, then $\overline{[I, A]} = \overline{[J, A]}$.

Proof. Let $\{e_\lambda\}$ be a quasi-central approximate identity for the ideal I in A . Since I is a closed ideal, clearly

$$\overline{[I, I]} \subseteq \overline{[I, A]} \subseteq I \cap \overline{[A, A]}.$$

For the reverse inclusions, we first show that $I \cap \overline{[A, A]} \subseteq \overline{[I, A]}$. Let $z \in I \cap \overline{[A, A]}$ and $\epsilon > 0$. Then, there exist $x_i, y_i \in A$, $1 \leq i \leq n$ such that $\|z - \sum_i [x_i, y_i]\| < \epsilon/3$. Note that $\sum_i [x_i e_\lambda, y_i] \in [I, A]$ for all λ , and

$$\begin{aligned} \|z - \sum_i [x_i e_\lambda, y_i]\| &\leq \|z - z e_\lambda\| + \|z e_\lambda + \sum_i x_i [e_\lambda, y_i] - \sum_i [x_i e_\lambda, y_i]\| + \|\sum_i x_i [e_\lambda, y_i]\| \\ &= \|z - z e_\lambda\| + \|z e_\lambda - \sum_i [x_i, y_i] e_\lambda\| + \sum_i \|x_i\| \|e_\lambda y_i - y_i e_\lambda\|. \end{aligned}$$

Thus, there exists an index λ_0 such that $\|z - \sum_i [x_i e_{\lambda_0}, y_i]\| < \epsilon$ implying that $z \in \overline{[I, A]}$.

For the remaining equality, it suffices to show that $[I, I]$ is dense in $\overline{[I, A]}$. Let $w \in \overline{[I, A]}$ and $\epsilon > 0$. Then there exist $u_i \in I, a_i \in A$, $1 \leq i \leq n$ such that $\|w - \sum_i [u_i, a_i]\| < \epsilon/3$. Clearly, $\sum_i [u_i, e_\lambda a_i] \in [I, I]$ and, as above, it is easily seen that

$$\|w - \sum_i [u_i, e_\lambda a_i]\| \leq \|w - e_\lambda w\| + \|e_\lambda w - e_\lambda \sum_i [u_i, a_i]\| + \sum_i \|u_i\| \|e_\lambda a_i - a_i e_\lambda\|,$$

implying that $[I, I]$ is dense in $\overline{[I, A]}$.

Since $\overline{[I, A]} \subseteq I$, by definition, $N(\overline{[I, A]}) \subseteq N(I)$ and if $x \in N(I)$, then $[x, A] \subseteq I \cap [A, A] \subseteq \overline{[I, A]}$ implying that $x \in N(\overline{[I, A]})$ and hence $N(\overline{[I, A]}) = N(I)$.

If A has no tracial functionals, then $\overline{[A, A]} = A$ and, therefore, $\overline{[I, A]} = I \cap \overline{[A, A]} = I$.

Finally, suppose the closed ideal $J := \overline{\text{Id}[I, A]}$ admits a quasi-central approximate identity, say, $\{f_\mu\}$. Since $J \subseteq I$, clearly $\overline{[J, A]} \subseteq \overline{[I, A]}$. Let $x \in I$ and $a \in A$. Then, $[x, a] \in J$, $[f_\mu x, a] \in [J, A]$ and

$$\begin{aligned} \|[f_\mu x, a] - [x, a]\| &\leq \|[f_\mu x, a] - f_\mu [x, a]\| + \|f_\mu [x, a] - [x, a]\| \\ &= \|f_\mu a - a f_\mu\| \|x\| + \|f_\mu [x, a] - [x, a]\| \rightarrow 0, \end{aligned}$$

implying that $[J, A]$ is dense in $[I, A]$ and hence $\overline{[I, A]} = \overline{[J, A]}$. \square

More generally, using a result by Robert [23], we shall show below that $\overline{[L, A]} = \overline{[\text{Id}[L, A], A]}$ for any closed Lie ideal L in an appropriate Banach algebra A , which generalizes [6, Theorem 5.27]. Robert, in [23], has given a simpler proof of [6, Theorem 5.27] avoiding von Neumann algebra tools.

Recall that a Banach algebra A is said to be semiprime if $I^2 = (0)$ implies $I = (0)$ for any closed ideal I . And a closed ideal I in A is said to be semiprime if the quotient Banach algebra A/I is semiprime. A C^* -algebra and all its closed ideals are easily seen to be semiprime.

LEMMA 4.3. *Let A be a Banach algebra whose every closed ideal possesses a left or a right approximate identity. Then A is semiprime and so are its closed ideals.*

Proof. Let I be a closed ideal in A such that $I^2 = (0)$. Let $x \in I$ and $\{e_\lambda\}$ be a right approximate identity in I . Then, $xe_\lambda = 0$ for all λ and as $xe_\lambda \rightarrow x$, we get $x = 0$ implying that $I = 0$. Thus, A is semiprime.

Next, for a closed ideal I in A , every closed ideal in A/I is of the form J/I for some closed ideal J in A containing I . If J admits a left or a right approximate identity, so does J/I . Therefore, every closed ideal in A/I admits a left or a right approximate identity and, as above, A/I is semiprime. \square

We will need the following observation by Br sar et al. [6, Proposition 5.2].

PROPOSITION 4.4. *Let L be a closed Lie ideal in a Banach algebra A and I_L denote the closed ideal generated by $[L, L]$, i.e., $I_L := \overline{\text{Id}[L, L]}$. If the Banach algebra A/I_L is semiprime or commutative, then $[L, A] \subseteq I_L$.*

The following mildly generalizes [6, Theorem 5.27] and a part of [23, Theorem 1.5].

THEOREM 4.5. *Let L be a closed Lie ideal in a Banach algebra A and I denote the closed ideal generated by $[L, A]$, i.e., $I := \overline{\text{Id}[L, A]}$. If all closed ideals of A containing I_L possess quasi-central approximate identities, then $I = I_L$ and*

$$(4.2) \quad \overline{[I, A]} = \overline{[L, A]}.$$

In particular, $\overline{[I, A]} \subseteq L \subseteq N(\overline{[I, A]})$. Moreover, if A has no tracial functionals, then $I \subseteq L \subseteq N(I)$, as well.

Proof. By Lemma 4.3, A/I_L is semiprime. So, by Proposition 4.4, $[L, A] \subseteq I_L$ implying that $I = I_L$. It is elementary to see that $\text{Id}[L, L] \subseteq L + L^2$ (see [23, Lemma 1.4]), so that $I \subseteq L + L^2$ and, therefore, $\overline{[I, A]} \subseteq \overline{[L + L^2, A]} = \overline{[L, A]}$, where the last equality follows from the easily verifiable fact that $[L^2, A] \subseteq [L, A]$ (see [23, (1.1)]).

On the other hand, since $I_L = I$, I contains a quasi-central approximate identity and, $\overline{[L, A]} \subseteq I \cap \overline{[A, A]} = \overline{[I, A]}$, by Lemma 4.2.

The remaining then follows again from Lemma 4.2. \square

COROLLARY 4.6. *Let A be a Banach algebra whose every closed ideal contains a quasi-central approximate identity and suppose $\mathcal{TF}(A) = \emptyset$. Then every non-central closed Lie ideal of A , i.e., $L \not\subseteq Z(A)$, contains a non-zero closed ideal.*

Theorem 4.5 partially answers a question of Br sar et al. [6, page 120] where they ask for suitable conditions in Banach $*$ -algebra setting so that a closed Lie ideal is closed

commutator equal to a closed ideal, and it also yields the following characterization of closed Lie ideals:

COROLLARY 4.7. *If every closed ideal in a Banach algebra A admits a quasi-central approximate identity, then a closed subspace L of A is a Lie ideal if and only if there exists a closed ideal I in A such that*

$$\overline{[I, A]} \subseteq L \subseteq N(\overline{[I, A]}).$$

The technique of Robert [23], based on a Theorem of Herstein [11], yields a stronger version of Theorem 4.5.

THEOREM 4.8. *Let L be a closed Lie ideal in a Banach algebra A , I denote the closed ideal generated by $[L, A]$ and M denote the closed Lie ideal $\overline{[L, A]}$. If all closed ideals of A containing $I_M := \overline{\text{Id}[M, M]}$ possess quasi-central approximate identities, then*

$$(4.3) \quad I = \overline{[L, A]} + \overline{[L, A]^2} = B([L, A]) \text{ and}$$

$$(4.4) \quad \overline{[I, A]} = \overline{[L, A]} = \overline{[[L, A], A]},$$

where $B([L, A])$ denotes the Banach subalgebra of A generated by $[L, A]$.

Proof. By Lemma 4.3, the quotient A/I_M is semiprime. So, the proof of the equalities $I = \overline{[L, A]} + \overline{[L, A]^2} = B([L, A])$ given by Robert in [23, Theorem 1.5 (i)] works verbatim.

Then, the inclusion $\overline{[I, A]} \subseteq \overline{[[L, A], A]} \subseteq \overline{[L, A]}$ is immediate. Since $I_M \subseteq I$, I contains a quasi-central approximate identity, so by Lemma 4.2, we have $\overline{[I, A]} = I \cap \overline{[A, A]}$ and, since $[L, A] \subseteq I \cap [A, A]$, the reverse inclusion follows. \square

By Lemma 4.2 and Theorem 2.1, Theorem 4.8 immediately yields the following:

COROLLARY 4.9. *Let A be a Banach algebra whose every closed ideal admits a quasi-central approximate identity. Then, for any closed ideal I in A , we have*

$$(4.5) \quad \overline{[I, I]} = \overline{[I, A]} = \overline{[I, [I, A]]} = I \cap \overline{[A, A]}.$$

In particular, if A is a C^* -algebra with no tracial states, then $\overline{[I, A]} = I$.

This yields the following generalization of [6, Corollary 5.26] and, using Lemma 4.2 and Corollary 4.9, the same proof works verbatim.

COROLLARY 4.10. *Let I (resp., L) be a closed ideal (resp., Lie ideal) in a Banach algebra A . If every closed ideal of A possesses a quasi-central approximate identity, then the following are equivalent:*

- (1) $\overline{[I, A]} \subseteq L \subseteq N(I)$.
- (2) $\overline{[I, A]} \subseteq L \subseteq N(\overline{[I, A]})$.
- (3) $\overline{[I, A]} = \overline{[L, A]}$.

4.1. Commutators of closed ideals in certain tensor products of C^* -algebras.

Apart from the usual spatial tensor product of C^* -algebras, we will also be interested in some tensor products which yield Banach algebras which are not necessarily C^* -algebras. As in [5, §2], a norm $\|\cdot\|_\alpha$ on the algebraic tensor product $A \otimes B$ of a pair of C^* -algebras A and B is said to be

- (1) a *sub-cross norm* if $\|a \otimes b\|_\alpha \leq \|a\| \|b\|$ for all $a \in A, b \in B$,
- (2) an *algebra norm* if $\|wz\|_\alpha \leq \|w\|_\alpha \|z\|_\alpha$ for all $w, z \in A \otimes B$, and

- (3) a *tensor norm* if $\|\cdot\|_\lambda \leq \|\cdot\|_\alpha \leq \|\cdot\|_\gamma$, where λ and γ are the Banach space injective and projective norms, respectively.

Clearly, $A \otimes^\alpha B$, the completion of $A \otimes B$ with respect to any algebra norm $\|\cdot\|_\alpha$, is a Banach algebra. Since $\|\cdot\|_\gamma$ is a cross norm, every tensor norm is, therefore, sub-cross.

The tensor products that we will be concerned with here include the C^* -minimal tensor product (\otimes^{\min}), the (operator space) Haagerup tensor product (\otimes^h), the operator space projective tensor product ($\widehat{\otimes}$) and the Banach space projective tensor product (\otimes^γ). We refer the reader to [8, 10] for their definitions and essential properties. All these norms are sub-cross algebra tensor norms and yield Banach algebras. In fact, for any pair of C^* -algebras, \otimes^γ (by definition) and $\widehat{\otimes}$ (by [12]) yield Banach $*$ -algebras whereas the natural involution is not isometric with respect to \otimes^h ([5]).

The following proposition is an immediate generalization of [3, Corollary 3.4] and yields examples of closed ideals with quasi-central approximate identities in Banach algebras which are not C^* -algebras.

PROPOSITION 4.11. *Let A and B be C^* -algebras and α be an algebra tensor norm. Let $\{I_i : 1 \leq i \leq n\}$ and $\{J_i : 1 \leq i \leq n\}$ be closed ideals in A and B , respectively. Then the closed ideal $K := \overline{\sum_i I_i \otimes J_i}^\alpha$ admits a quasi-central approximate identity in $A \otimes^\alpha B$.*

Proof. By [3, Lemma 3.3], the closure of a finite sum of closed ideals containing quasi-central approximate identities in a Banach algebra also contains a quasi-central approximate identity. And since, $\overline{\sum_i I_i \otimes J_i}^\alpha = \overline{\sum_i \overline{I_i} \otimes \overline{J_i}}^{\alpha^\alpha}$, it is enough to show that an arbitrary product ideal $\overline{I \otimes J}^\alpha$, for ideals I and J in A and B , respectively, admits a quasi-central approximate identity in $A \otimes^\alpha B$.

Let $\{e_\lambda : \lambda \in \Lambda\}$ and $\{f_\gamma : \gamma \in \Gamma\}$ be quasi-central approximate identities for I and J in A and B , respectively ([2, Theorem 1]), with $K = \sup_\lambda \|e_\lambda\|$ and $L = \sup_\gamma \|f_\gamma\|$. The set $\Lambda \times \Gamma$ inherits a directed structure via the partial ordering

$$(\lambda_1, \gamma_1) \leq (\lambda_2, \gamma_2) \text{ if and only if } \lambda_1 \leq \lambda_2 \text{ and } \gamma_1 \leq \gamma_2.$$

Let $e_{(\lambda, \gamma)} := e_\lambda$ and $f_{(\lambda, \gamma)} := f_\gamma$ for all $(\lambda, \gamma) \in \Lambda \times \Gamma$, and set $z_\mu = e_\mu \otimes f_\mu$ for all $\mu \in \Lambda \times \Gamma$. Clearly, $\{e_\mu : \mu \in \Lambda \times \Gamma\}$ and $\{f_\mu : \mu \in \Lambda \times \Gamma\}$ are quasi-central approximate identities for I and J , respectively. We show that $\{z_\mu : \mu \in \Lambda \times \Gamma\}$ is a quasi-central approximate identity for $\overline{I \otimes J}^\alpha$ in $A \otimes^\alpha B$. Since α is a sub-cross norm, $\{z_\mu\}$ is uniformly bounded. Let $x = \sum_i u_i \otimes v_i \in I \otimes J$ and $y = \sum_i a_i \otimes b_i \in A \otimes B$. Then,

$$\begin{aligned} \|xz_\mu - x\|_\alpha &\leq \sum_i \|(u_i e_\mu - u_i) \otimes v_i f_\mu\|_\alpha + \|u_i \otimes (v_i f_\mu - v_i)\|_\alpha \\ &\leq \sum_i L \|u_i e_\mu - u_i\| \|v_i\| + \|u_i\| \|v_i f_\mu - v_i\| \rightarrow 0, \end{aligned}$$

likewise, $\|z_\mu x - x\|_\alpha \rightarrow 0$, and

$$\begin{aligned} \|yz_\mu - z_\mu y\| &\leq \sum_i \|a_i e_\mu - e_\mu a_i\| \|b_i f_\mu\| + \sum_i \|e_\mu a_i\| \|b_i f_\mu - f_\mu b_i\| \\ &\leq \sum_i L \|a_i e_\mu - e_\mu a_i\| \|b_i\| + \sum_i K \|a_i\| \|b_i f_\mu - f_\mu b_i\| \rightarrow 0. \end{aligned}$$

Since $I \otimes J$ (resp., $A \otimes B$) is dense in $\overline{I \otimes J}^\alpha$ (resp., $A \otimes^\alpha B$), it follows that $\lim_\mu \|xz_\mu - x\|_\alpha = \lim_\mu \|z_\mu x - x\|_\alpha = \lim_\mu \|yz_\mu - z_\mu y\| = 0$ for all $x \in \overline{I \otimes J}^\alpha$ and $y \in A \otimes^\alpha B$. \square

REMARK 4.12. Note that in the above theorem, we have actually proved that, if I and J are ideals (not necessarily closed) in C^* -algebras A and B , then the (algebraic) product ideal $I \otimes J$ admits a quasi-central approximate identity in $A \otimes^\alpha B$.

We can now easily deduce the following:

COROLLARY 4.13. *Let A and B be C^* -algebras and α be an algebra tensor norm. Let $\{J_i : 1 \leq i \leq n\}$ and $\{K_i : 1 \leq i \leq n\}$ be closed ideals in A and B , respectively. Then,*

$$\overline{[I, I]} = \overline{[I, A \otimes^\alpha B]} = I \cap \overline{[A \otimes^\alpha B, A \otimes^\alpha B]}$$

if I is a closed ideal in $A \otimes^\alpha B$ of any of the following form:

- (1) $I = \overline{\sum_i J_i \otimes K_i}^\alpha$.
- (2) $I = \sum_i J_i \otimes^\alpha K_i$ and α is either the Haagerup norm or the operator space projective norm.
- (3) I is any closed ideal in $A \otimes^\alpha B$, A contains only finitely many closed ideals and α is either the Haagerup norm or the operator space projective norm.
- (4) I is any ideal in $A \otimes^\alpha B$ and α is any C^* -tensor norm.

Proof. (1) is immediate from Proposition 4.11 and Lemma 4.2.

(2): By [3, Theorem 3.8] and [14, Proposition 3.2], $\sum_i J_i \otimes^\alpha K_i$ is a closed ideal in $A \otimes^\alpha B$.

(3): By [3, Theorem 5.3] and [16, Theorem 3.4], every closed ideal in $A \otimes^\alpha B$ is a finite sum of product ideals.

(4) follows from the fact that every ideal in a C^* -algebra admits a quasi-central approximate identity ([1, 2]). □

REMARK 4.14. If a C^* -algebra A ($\not\cong \mathbb{C}$) contains only finitely many closed ideals, then for any C^* -algebra B ($\not\cong \mathbb{C}$), $A \otimes^h B$ or $A \widehat{\otimes} B$ is a Banach algebra which is not a C^* -algebra ([5, Theorem 1]) and, as seen above, its every closed ideal possesses a quasi-central approximate identity.

PROPOSITION 4.15. *Let A and B be C^* -algebras and suppose B is unital. Then, every non-central closed Lie ideal in $A \otimes^\alpha B$ contains a non-zero closed ideal in the following cases:*

- (1) A has no tracial states and α is any C^* -norm.
- (2) A has no tracial functionals, A contains only finitely many closed ideals and α is either the Haagerup norm or the operator space projective norm.

Proof. (1): Since B is unital, for any C^* -norm α , $A \subseteq A \otimes^\alpha B$ as a C^* -subalgebra, so $A \otimes^\alpha B$ does not have any tracial states and, therefore, the assertion holds by Theorem 4.5 and Corollary 4.9.

(2): Since α is a cross norm (see [8]), $A \ni a \mapsto a \otimes 1 \in A \otimes^\alpha B$ is an isometric homomorphism; so, the Banach algebra $A \otimes^\alpha B$ admits no tracial functionals. The rest is then taken care of by Corollary 4.13(3) and Corollary 4.6. □

THEOREM 4.16. *Let A and B be simple, unital C^* -algebras and suppose one of them admits no tracial functionals. If α is either the Haagerup norm, the operator space projective norm or the C^* -minimal norm, then the only closed Lie ideals of $A \otimes^\alpha B$ are $\{0\}$, $\mathbb{C}(1 \otimes 1)$ and $A \otimes^\alpha B$ itself.*

Proof. If \otimes^α is \otimes^h or \otimes^{\min} , it is known ([3, Theorem 2.13] and [25, Corollary 1]) that $Z(A \otimes^\alpha B) = Z(A) \otimes^\alpha Z(B)$. And, by [13, Theorem 3], the algebraic isomorphism $Z(A) \otimes Z(B) \cong Z(A \otimes B)$ extends to an algebraic isomorphism (not necessarily isometric) between $Z(A) \widehat{\otimes} Z(B)$ and $Z(A \widehat{\otimes} B)$. So, in all three cases, we obtain $Z(A \otimes^\alpha B) = \mathbb{C}(1 \otimes 1)$. In particular, the only central Lie ideals of $A \otimes^\alpha B$ are $\{0\}$ and $\mathbb{C}(1 \otimes 1)$.

The C^* -algebra $A \otimes^{\min} B$ is simple (see [24, Corollary IV.4.21]). And, by [3, Theorem 5.1] and [15, Theorem 3.7], the Banach algebras $A \otimes^h B$ and $A \widehat{\otimes} B$ are topologically simple. So, by Proposition 4.15, $A \otimes^\alpha B$ is its only non-central closed Lie ideal. \square

We conclude this section with the following:

THEOREM 4.17. *Let H be an infinite dimensional separable Hilbert space. If α is either the Haagerup norm, the operator space projective norm or the C^* -minimal norm, then the only non-zero central Lie ideal of $B(H) \otimes^\alpha B(H)$ is $\mathbb{C}(1 \otimes 1)$ and every non-central closed Lie ideal of $B(H) \otimes^\alpha B(H)$ contains the product ideal $K(H) \otimes^\alpha K(H)$.*

Proof. As in Theorem 4.16, we obtain $Z(B(H) \otimes^\alpha B(H)) = \mathbb{C}(1 \otimes 1)$.

Now, let L be a non-central closed Lie ideal in $B(H) \otimes^\alpha B(H)$. By a theorem of Halmos, every bounded operator on H is a sum of two commutators, so $B(H)$ does not admit any tracial functionals. Thus, L must contain a non-zero closed ideal by Proposition 4.15. By [3, Proposition 4.5 and Corollary 4.6] and [15, Proposition 3.6], every non-zero closed ideal of $B(H) \otimes^\alpha B(H)$ contains an elementary tensor, say, $a \otimes b$. So, $K(H)$ being the only non-trivial closed ideal in $B(H)$, $K(H) \otimes^\alpha K(H)$ must be contained in $\overline{\text{Id}\{a\}} \otimes^\alpha \overline{\text{Id}\{b\}}$. In other words, $K(H) \otimes^\alpha K(H)$ is the unique minimal closed ideal which is contained in every non-zero closed ideal of $B(H) \otimes^\alpha B(H)$. Therefore, in all cases, L must contain the product ideal $K(H) \otimes^\alpha K(H)$. \square

5. CLOSED LIE IDEALS OF $A \otimes^{\min} C(X)$

Let X be a compact Hausdorff space and A be a unital C^* -algebra. It is well known ([10, § 5]) that the canonical map $A \otimes^{\min} C(X) \ni a \otimes f \mapsto f(\cdot)a \in C(X, A)$ extends to a unital C^* -isomorphism. We will be using this fact and its consequences in the following observations.

We first recall, from [17], certain naturally arising closed ideals and closed Lie ideals of $A \otimes^{\min} C(X)$. Some of the proofs were not given in [17]. For the sake of completeness and convenience, we provide outlines of those proofs in bigger generality. The following folklore observation for a UHF C^* -algebra was used in [17, Theorem 3.1]. We include the details for a more general situation.

PROPOSITION 5.1. *Let X be a compact Hausdorff space and A be a topologically simple C^* -algebra. Then every closed ideal in $C(X, A)$ is of the form $\{f \in C(X, A) : f(s) = 0 \text{ for all } s \in F\}$ for some closed subset F of X .*

Proof. Let I be a closed ideal in $C(X, A)$. From Theorem 3.1 and the well known fact that every closed ideal in $C(X)$ is of the form $J(F) := \{f \in C(X) : f(s) = 0 \text{ for all } s \in F\}$ for some closed subset F of X , I corresponds to the ideal $A \otimes^{\min} J(F)$ in $A \otimes^{\min} C(X)$. It is enough to show that $I = \tilde{J}(F)$ where $\tilde{J}(F) := \{f \in C(X, A) : f(s) = 0 \text{ for all } s \in F\}$, which is clearly a closed ideal in $C(X, A)$.

Clearly $I \subseteq \tilde{J}(F)$. To obtain the equality we just need to show that I is dense in $\tilde{J}(F)$. Let $f \in \tilde{J}(F)$ and $\epsilon > 0$. For each $a \in A$, consider the open ball $B_\epsilon(a) := \{x \in A : \|x - a\| <$

$\epsilon\}$ and the punctured open ball $B_\epsilon^\times(a) := \{x \in A \setminus \{0\} : \|x - a\| < \epsilon\}$. The collection $\{f^{-1}(B_\epsilon^\times(f(x))) : x \in X \setminus F\} \cup \{f^{-1}(B_\epsilon(0))\}$ is an open cover of X . Fix a finite subcover, say, $\{f^{-1}(B_\epsilon^\times(f(x_i))) : 1 \leq i \leq n\} \cup \{f^{-1}(B_\epsilon(0))\}$. Since X is compact and Hausdorff, there exists a partition of unity $\{\varphi_i : 0 \leq i \leq n\}$ such that $\text{supp}(\varphi_i) \subseteq U_i := f^{-1}(B_\epsilon^\times(f(x_i)))$ for $1 \leq i \leq n$ and $\text{supp}(\varphi_0) \subseteq U_0 := f^{-1}(B_\epsilon(0))$. Then, $\varphi_i \in J(F)$ for all $1 \leq i \leq n$, so that $\sum_{i=1}^n f(x_i) \otimes \varphi_i \in A \otimes J(F)$. Fix an $x_0 \in F$. Then, for each $x \in X$, we have

$$\begin{aligned} \|f(x) - \sum_{i=1}^n \varphi_i(x) f(x_i)\| &= \|f(x) \sum_{i=0}^n \varphi_i(x) - \sum_{i=0}^n \varphi_i(x) f(x_i)\| \\ &\leq \sum_{i=0}^n \|f(x) - f(x_i)\| \varphi_i(x) \\ &= \sum_{i: x \in U_i} \|f(x) - f(x_i)\| \varphi_i(x) \\ &< \epsilon. \end{aligned}$$

In particular, $\|f - \sum_{i=1}^n f(x_i) \varphi_i\| < \epsilon$, implying that I is dense in $\tilde{J}(F)$. \square

For a Lie ideal L in a unital C^* -algebra A , a subspace S and an ideal J in $C(X)$, it is easily seen that $L \otimes J + \mathbb{C}1 \otimes S$ is a Lie ideal in $A \otimes C(X)$. Note that, if L , J and S are closed, then it is not clear whether the sum $\overline{L \otimes J} + \mathbb{C}1 \otimes S$ is closed or not. Marcoux, in [17, Theorem 3.1], had shown that for a UHF C^* -algebra A , the sum $\overline{sl(A) \otimes J} + \mathbb{C}1 \otimes S$ is always closed. Exploiting Marcoux's technique, we prove the same in a more general setting. Before that, we first make the following observation:

LEMMA 5.2. *Let X be a compact Hausdorff space and A be a C^* -algebra with $\mathcal{T}(A) \neq \emptyset$. Then, for any closed set F in X , the closed Lie ideal $L(F) := \overline{sl(A) \otimes J(F)}$ in $A \otimes^{\min} C(X)$ corresponds to the closed Lie ideal*

$$\tilde{L}(F) := \{f \in C(X, A) : f(s) = 0 \text{ for all } s \in F \text{ and } \varphi \circ f = 0 \text{ for all } \varphi \in \mathcal{T}(A)\}$$

in $C(X, A)$.

Proof. Clearly, under the canonical $*$ -isomorphism between $A \otimes^{\min} C(X)$ and $C(X, A)$, the closed Lie ideal $L(F)$ is mapped onto a closed Lie ideal in $\tilde{L}(F)$. It just remains to show that the image is dense in $\tilde{L}(F)$. Let $f \in \tilde{L}(F)$ and $\epsilon > 0$. Then, $f \in \tilde{J}(F)$, and as in the proof of Proposition 5.1, there exist finite sets $\{x_1, \dots, x_n\} \subseteq X \setminus F$ and $\{\varphi_1, \dots, \varphi_n\} \subseteq C(X)$ such that $\varphi_i(F) = \{0\}$ for all $1 \leq i \leq n$ and $\|f - \sum_{i=1}^n f(x_i) \varphi_i\| < \epsilon$. Since $sl(A) = \bigcap_{\varphi \in \mathcal{T}(A)} \ker(\varphi)$ and $f \in \tilde{L}(F)$, it readily follows that $\sum_{i=1}^n f(x_i) \otimes \varphi_i \in sl(A) \otimes J(F)$ and we are done. \square

An adaptation of Marcoux's proof, gives us the following generalization of the sufficient condition of [17, Theorem 3.1].

PROPOSITION 5.3. *Let X be a compact Hausdorff space and A be a unital C^* -algebra. If $\mathcal{T}(A) \neq \emptyset$, then a subspace of the form $L = \overline{sl(A) \otimes J} + \mathbb{C}1 \otimes S$, where J is a closed ideal and S is a closed subspace in $C(X)$, is a closed Lie ideal in the C^* -algebra $A \otimes^{\min} C(X)$.*

Proof. It is easy to verify that $L = \overline{sl(A) \otimes J} + \mathbb{C}1 \otimes S$ is a Lie ideal. We only need to show that L is closed. Now, let $\{f_n\}$ be a sequence in L converging to some f in $A \otimes^{\min} C(X)$. Decompose $f_n = g_n + h_n$, where $g_n \in \overline{sl(A) \otimes J}$ and $h_n \in \mathbb{C}1 \otimes S$. Since $A \otimes^{\min} C(X)$

is isometrically isomorphic to $C(X, A)$, we can assume that $\varphi \circ f_n, \varphi \circ f \in C(X)$ for all $\varphi \in \mathcal{T}(A)$ and $n \geq 1$.

Clearly $\{\varphi \circ f_n\}$ is uniformly convergent to $\varphi \circ f$ for all $\varphi \in \mathcal{T}(A)$. Since $g_n \in \overline{sl(A) \otimes J}$, we have $\varphi \circ f_n = \varphi \circ h_n$, so that $\varphi \circ h_n$ also converges uniformly to $\varphi \circ f$ for all $\varphi \in \mathcal{T}(A)$.

Since $h_n \in \mathbb{C}1 \otimes S$, there exists a $\mu_n \in C(X)$ such that $h_n(x) = \mu_n(x)1$ for all $x \in X$; so that $\varphi \circ h_n = \mu_n$ for all $\varphi \in \mathcal{T}(A)$ and $n \geq 1$. This implies that $\{h_n\}$ is Cauchy and hence converges uniformly to some h in $A \otimes^{\min} C(X)$. Since S is closed, $\mathbb{C}1 \otimes S$ is closed and we have $h \in \mathbb{C}1 \otimes S$.

Finally, $\{g_n\}$ converges uniformly to $f - h = g$ (say) in $C(X, A)$. Since $\varphi \circ g_n = 0$ for all $\varphi \in \mathcal{T}(A)$ and $n \geq 1$, we have $\varphi \circ g = 0$ for all $\varphi \in \mathcal{T}(A)$. Also, if $X_J := \{x \in X : f(x) = 0 \text{ for all } f \in J\}$, then $g_n(X_J) = \{0\}$ for all n and hence $g(X_J) = \{0\}$ as well. Therefore, by Lemma 5.2, $g \in \overline{sl(A) \otimes J}$ and L is closed. \square

In the reverse direction, as yet another application of the C^* -isomorphism between $A \otimes^{\min} C(X)$ and $C(X, A)$, we will have two instances to appeal to the following observation (from [17, Theorem 3.1]):

LEMMA 5.4. *Let X be a compact Hausdorff space and A be a simple unital C^* -algebra. For any closed ideal I in $A \otimes^{\min} C(X)$, we have*

$$N(I) = I + \mathbb{C}1 \otimes C(X).$$

Proof. By Theorem 3.1, I is of the form $A \otimes^{\min} J$ for some closed ideal J in $C(X)$. Clearly, $A \otimes^{\min} J + \mathbb{C}1 \otimes C(X) \subseteq N(A \otimes^{\min} J)$. And if $f \in N(A \otimes^{\min} J) \subseteq C(X, A)$, then $[f, g] \in A \otimes^{\min} J$ for all $g \in C(X, A)$. In particular, for each $a \in A$, if $f_a \in C(X, A)$ denotes the constant function taking the value a , then $f(x)a - af(x) = 0$ for all $x \in F$, where F is the closed set in X that determines the closed ideal J , i.e., $J(F) = J$. Thus, $f(x) \in \mathcal{Z}(A) = \mathbb{C}1$ for all $x \in F$. If $g := f|_F$, then after identifying $C(X)$ with $\mathbb{C}1 \otimes C(X)$, by Tietze's Extension Theorem, g can be extended to a scalar-valued map \tilde{g} on X so that $\tilde{g} \in \mathbb{C}1 \otimes C(X)$. Since $f(x) - \tilde{g}(x) = 0$ for all $x \in F$, we have $f - \tilde{g} \in A \otimes^{\min} J$, so that $f \in A \otimes^{\min} J + \mathbb{C}1 \otimes C(X)$. \square

LEMMA 5.5. *Let X be a compact Hausdorff space, A be a simple unital C^* -algebra and J be a closed ideal in $C(X)$. Then,*

- (1) $\overline{[A \otimes^{\min} C(X), A \otimes^{\min} J]} = \begin{cases} \overline{sl(A) \otimes J} & \text{if } \mathcal{T}(A) \neq \emptyset, \text{ and} \\ A \otimes^{\min} J & \text{if } \mathcal{T}(A) = \emptyset. \end{cases}$
- (2) *If A admits a unique tracial state and L is a closed subspace of $A \otimes^{\min} C(X)$ satisfying*

$$\overline{[A \otimes^{\min} C(X), A \otimes^{\min} J]} \subseteq L \subseteq N(A \otimes^{\min} J),$$

then L is a closed Lie ideal and is of the form $L = \overline{sl(A) \otimes J} + \mathbb{C}1 \otimes S$ for some closed subspace S in $C(X)$.

Proof. (1) By Lemma 4.2, we have $\overline{[A \otimes^{\min} C(X), A \otimes^{\min} J]} = \overline{[A \otimes^{\min} J, A \otimes^{\min} J]}$ and it is easily verified that $\overline{[A \otimes^{\min} J, A \otimes^{\min} J]} = \overline{[A, A]} \otimes J$. Therefore, by Theorem 2.1, we obtain the desired forms for $\overline{[A \otimes^{\min} C(X), A \otimes^{\min} J]}$.

(2) By Lemma 4.2, we also have $N(A \otimes^{\min} J) = N(\overline{[A \otimes^{\min} C(X), A \otimes^{\min} J]})$, and therefore L is a closed Lie ideal. The fact that L must be of above form follows on the lines of a part of the proof of [17, Theorem 3.1]. We, therefore, just mention the steps involved and omit the details:

From (1), we see that $\overline{sl(A) \otimes J} \subseteq L$. By Lemma 5.4, we have $N(A \otimes^{\min} J) = A \otimes^{\min} J + \mathbb{C}1 \otimes C(X)$, and since $\mathcal{T}(A)$ is a singleton, we also have $A = \mathbb{C}1 \oplus sl(A)$. Using Proposition 5.1 and Lemma 5.2, one then deduces that, in fact, $N(A \otimes^{\min} J) = \overline{sl(A) \otimes J} + \mathbb{C}1 \otimes C(X)$. And, therefore, since L is a closed subspace satisfying

$$\overline{sl(A) \otimes J} \subseteq L \subseteq \overline{sl(A) \otimes J} + \mathbb{C}1 \otimes C(X),$$

by Proposition 5.3, we must have $L = \overline{sl(A) \otimes J} + \mathbb{C}1 \otimes S$ for some closed subspace S of $C(X)$. \square

Note that a subspace of the form $L = \overline{A \otimes J + \mathbb{C}1 \otimes S}$ where J is a closed ideal and S is a subspace of $C(X)$ is clearly a closed Lie ideal of the C^* -algebra $A \otimes^{\min} C(X)$. The crux of the next theorem is that all closed Lie ideals of $A \otimes^{\min} C(X)$ arise as in Proposition 5.3, the first part of which is a generalization of the necessary condition of [17, Theorem 3.1].

THEOREM 5.6. *Let X be a compact Hausdorff space and A be a simple unital C^* -algebra with at most one tracial state. Then a subspace L of $A \otimes^{\min} C(X)$ is a closed Lie ideal if and only if*

$$L = \begin{cases} \overline{sl(A) \otimes J} + \mathbb{C}1 \otimes S & \text{if } \mathcal{T}(A) \neq \emptyset, \text{ and} \\ \overline{A \otimes J + \mathbb{C}1 \otimes S} & \text{if } \mathcal{T}(A) = \emptyset, \end{cases}$$

for some closed ideal J and closed subspace S in $C(X)$.

Proof. We just need to prove the only if part in both cases. Let L be a closed Lie ideal in $A \otimes^{\min} C(X)$.

(1) Suppose $\mathcal{T}(A) \neq \emptyset$. Since $A \otimes^{\min} C(X)$ is a C^* -algebra and A is simple, by Corollary 4.7 and Theorem 3.1, there exists a closed ideal $I = A \otimes^{\min} J$ in $A \otimes^{\min} C(X)$ for some closed ideal J in $C(X)$, such that

$$\overline{[A \otimes^{\min} C(X), A \otimes^{\min} J]} \subseteq L \subseteq N(A \otimes^{\min} J).$$

L then has the required form by Lemma 5.5.

(2) Suppose A has no tracial states. Since $A \otimes^{\min} C(X)$ is a C^* -algebra, and A embeds in $A \otimes^{\min} C(X)$ as a C^* -subalgebra, $A \otimes^{\min} C(X)$ also does not admit any tracial state. Therefore, since A is simple, by Corollary 4.7, Corollary 4.9 and Theorem 3.1, there exists a closed ideal $I = A \otimes^{\min} J$ of $A \otimes^{\min} C(X)$ for some closed ideal J of $C(X)$, such that

$$A \otimes^{\min} J \subseteq L \subseteq N(A \otimes^{\min} J).$$

Again, as in (1), by Lemma 5.4, we see that $N(A \otimes^{\min} J) = A \otimes^{\min} J + \mathbb{C}1 \otimes C(X)$. Now, $A \otimes^{\min} J$ being closed, so is $N(A \otimes^{\min} J)$ implying that $A \otimes^{\min} J + \mathbb{C}1 \otimes C(X)$ is closed. In particular, L must be of the form

$$L = \overline{A \otimes J + \mathbb{C}1 \otimes S}$$

for some closed subspace S in $C(X)$. \square

6. CLOSED LIE IDEALS OF $A \otimes^{\alpha} K(H)$

In this section, we will analyze the Lie ideals of the tensor product spaces $A \otimes^h K(H)$ and $A \widehat{\otimes} K(H)$. For this, we need an auxillary result from Brešar et. al ([6]). For the sake of completeness, we include a short discussion on the pre-requisites. Throughout this section, H will be assumed to be a seperable Hilbert space.

Given any unit vector e in a Hilbert space H , one considers the rank one orthogonal projection $p_e : H \rightarrow H$ given by $p_e(x) = \langle x, e \rangle e, x \in H$. Then, for any finite orthonormal

system $\mathcal{E} = \{e_i : 1 \leq i \leq n\}$ in a Hilbert space H , consider the orthogonal projection $p_{\mathcal{E}} := \sum_i p_{e_i}$ and the completely positive maps $s_{\mathcal{E}}, t_{\mathcal{E}} : K(H) \rightarrow K(H)$ given by $s_{\mathcal{E}}(x) = \sum_i p_{e_i} x p_{e_i}$ and $t_{\mathcal{E}}(x) = p_{\mathcal{E}} x p_{\mathcal{E}}$ for $x \in K(H)$.

Clearly $t_{\mathcal{E}}$ is a complete contraction on $K(H)$ and the same is true about $s_{\mathcal{E}}$, which can be seen as follows.

LEMMA 6.1. *For every finite orthonormal system $\mathcal{E} = \{e_i : 1 \leq i \leq n\}$ in a Hilbert space H , $s_{\mathcal{E}}$ is a complete contraction on $K(H)$.*

Proof. It is enough to show that the map $B(H) \ni x \mapsto \sum_i p_{e_i} x p_{e_i} \in B(H)$ is a complete contraction. Consider the orthogonal decomposition $H = \oplus_{i=1}^n p_{e_i} H$. Then, for each $x \in B(H)$, $\sum_i p_{e_i} x p_{e_i}$ corresponds to the $n \times n$ diagonal matrix operator $\text{diag}(p_{e_1} x p_{e_1}, \dots, p_{e_n} x p_{e_n})$. Therefore, $\|\sum_i p_{e_i} x p_{e_i}\| \leq \|x\|$ for all $x \in B(H)$. Now, for $[x_{ij}] \in M_n(B(H))$, we have

$$\varphi^{(n)}([x_{ij}]) = [\varphi(x_{ij})] = \sum_i Q_i [x_{ij}] Q_i$$

where $Q_i := \text{diag}(p_{e_i}, p_{e_i}, \dots, p_{e_i})$ is a projection in $M_n(B(H)) = B(H^{(n)})$. So, as above $\varphi^{(n)}$ is a contraction on $B(H^{(n)}) = M_n(B(H))$ for all $n \geq 1$. \square

A tensor norm α is said to be *completely positive uniform* if for every completely positive map $T_i : A_i \rightarrow B_i$, $i = 1, 2$, the canonical linear map $T_1 \otimes T_2 : A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$ has a continuous extension $T_1 \otimes^{\alpha} T_2 : A_1 \otimes^{\alpha} A_2 \rightarrow B_1 \otimes^{\alpha} B_2$ satisfying $\|T_1 \otimes^{\alpha} T_2\| \leq \|T_1\| \|T_2\|$.

Many known tensor norms are completely positive uniform including the C^* -injective norm $\|\cdot\|_{\min}$, the C^* -projective norm $\|\cdot\|_{\max}$, the Haagerup norm $\|\cdot\|_h$, the operator space projective norm $\|\cdot\|_{\wedge}$ and the Banach space injective and projective norms $\|\cdot\|_{\lambda}$ and $\|\cdot\|_{\gamma}$ - see [5, 8, 24].

The following is an analogue of [6, Corollary 5.22] and follows directly from their deep result ([6, Theorem 5.15]) which involves some serious algebraic techniques.

PROPOSITION 6.2. *Let A be a unital C^* -algebra and α be a completely positive uniform algebra tensor norm. Then a closed subspace of the Banach algebra $A \otimes^{\alpha} K(H)$ is a Lie ideal if and only if it is a closed ideal and has the form $\overline{I \otimes K(H)}^{\alpha}$ for some closed ideal I in A .*

Proof. Let \mathcal{E} be a finite orthonormal system in H . Since α is completely positive uniform, and $s_{\mathcal{E}}, t_{\mathcal{E}}$ are completely positive and completely contractive, the maps $S_{\mathcal{E}} := \text{id} \otimes s_{\mathcal{E}}$, $T_{\mathcal{E}} := \text{id} \otimes t_{\mathcal{E}} : A \otimes F(H) \rightarrow A \otimes F(H)$ extend continuously on $A \otimes^{\alpha} K(H)$ and satisfy $\|S_{\mathcal{E}}\|, \|T_{\mathcal{E}}\| \leq 1$. Therefore, by [6, Theorem 5.15], every closed Lie ideal in $A \otimes^{\alpha} K(H)$ is a closed ideal and has the form $\overline{I \otimes K(H)}^{\alpha}$ for some closed ideal I of A . \square

COROLLARY 6.3. *If A is a unital C^* -algebra and α is either the Haagerup norm or the operator space projective norm, then any closed Lie ideal of $A \otimes^{\alpha} K(H)$ is precisely of the form $I \otimes^{\alpha} K(H)$ for some closed ideal I in A .*

Proof. The Haagerup norm $\|\cdot\|_h$ and the operator space projective norm $\|\cdot\|_{\wedge}$ are completely positive uniform algebra tensor norms ([5, Proposition 2] and [8, Theorem 3.2]). The assertion made in the statement then follows from Proposition 6.2, the injectivity of \otimes_h and the fact that $I \widehat{\otimes} K(H) \cong \overline{I \otimes K(H)}^{\alpha} \subseteq A \widehat{\otimes} K(H)$ ([12, Theorem 5]). \square

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